Standard forms and entanglement engineering of multimode Gaussian states under local operations

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Abstract. We investigate the action of local unitary operations on multimode (pure or mixed) Gaussian states and single out the minimal number of locally invariant parametres which completely characterise the covariance matrix of such states. For pure Gaussian states, central resources for continuous-variable quantum information, we investigate separately the parametre reduction due to the additional constraint of global purity, and the one following by the local-unitary freedom. Counting arguments and insights from the phase-space Schmidt decomposition and in general from the framework of symplectic analysis, accompany our description of the standard form of pure n-mode Gaussian states. In particular we clarify why only in pure states with $n \leq 3$ modes all the direct correlations between position and momentum operators can be set to zero by local unitary operations. For any n, the emerging minimal set of parametres contains complete information about all forms of entanglement in the corresponding states. An efficient state engineering scheme (able to encode direct correlations between position and momentum operators as well) is proposed to produce entangled multimode Gaussian resources, its number of optical elements matching the minimal number of locally invariant degrees of freedom of general pure n-mode Gaussian states. Finally, we demonstrate that so-called "block-diagonal" Gaussian states, without direct correlations between position and momentum, are systematically less entangled, on average, than arbitrary pure Gaussian states.

1. Prologue

Entanglement between the subsystems of composite quantum systems is arguably one of the most radical features of quantum mechanics, the one invoking a dramatic departure from classical principles [1]. This is probably one of the reasons why a fully satisfactory understanding and characterisation of such a feature in the most general setting is still lacking. Accordingly, the task of developing a comprehensive theoretical framework to qualify and quantify multipartite entanglement stands as a major issue to be achieved in quantum physics. A novel insight into the role of entanglement in the description of quantum systems has been gained through the quantum information perspective, mostly focusing on the usefulness of entanglement, rather than on its mathematical characterization. In these years, quantum entanglement has turned from a paradoxical concept into a physical resource allowing for the encoding, manipulation, processing and distribution of information in ways forbidden by the laws of classical physics. In this respect, entanglement between canonically conjugate continuous variables (CV) of infinite-dimensional systems, like harmonic oscillators, light modes and atomic ensembles, has emerged as a versatile and powerful resource [2]. In particular, multimode Gaussian states have been proven useful for a wide range of implementations in CV quantum information processing [3], and advances in the characterisation of their bipartite and multipartite entanglement have recently been recorded [4]. In experiments, one typically aims at preparing pure states, with the highest possible entanglement, even though unavoidable losses and thermal noises will affect the purity of the engineered resources, and hence the efficiency of the realised protocols [5]. It is therefore important to understand the structure of correlations in pure Gaussian states, and to provide 'economical' schemes to produce such states in the lab with minimal elements, thus reducing the possibility of accumulating errors and unwanted noise.

Gaussian states of CV systems are special in that they are completely specified by the first and second moments of the canonical bosonic operators. However, this already reduced set of parametres (compared to a true infinite-dimensional one needed to specify an arbitrary non-Gaussian CV state) contains many redundant degrees of freedom which have no effect on the entanglement. A basic property of multipartite entanglement is in fact its invariance under unitary operations performed locally on the subsystems. To describe entanglement efficiently, is thus natural to lighten quantum systems of the unnecessary degrees of freedom adjustable by local unitaries (LUs), and to classify states according to *standard forms* representative of LU equivalence classes [6]. When applied to Gaussian states of n modes, the freedom arising from the LU invariance immediately rules out the vector of first moments, which can be arbitrarily adjusted by local displacements in phase space (LUs on the Hilbert spaces) and thus made null without any loss of generality. One is then left with the $2n(2n+1)/2$ real parametres constituting the symmetric covariance matrix (CM) of the second moments (rigorously defined in the following).

In this paper, we study the action of LU operations on a general CM of a multimode Gaussian state. We compute the minimal number of parametres which completely characterise Gaussian states, up to LUs. The set of such parametres will contain *complete* information about any form of bipartite or multipartite entanglement in the corresponding Gaussian states. We give accordingly the standard form of the CM of a (generally mixed) n-mode Gaussian state. We then focus on pure states, the preferred resources for CV quantum communication and information processing, and study how the additional constraint of global purity leads to a further reduction of the minimal set of LU invariant parametres. We interpret those degrees of freedom in terms of correlations between the canonical operators of the various modes, and discuss how to engineer pure n-mode Gaussian states starting from a two-mode

squeezed state and $n-2$ single-mode squeezed beams, via passive operations only. Our results generalise the classification of Ref. [7], where the standard form of n-mode pure Gaussian states with no correlations between position ($\hat{x}$) and momentum ($\hat{p}$) operators was given, together with an optimal scheme to engineer such "block-diagonal" resources (employed in most CV quantum information protocols) in an optical setting. In this respect, we show that nonzero $\hat{x}$-$\hat{p}$ correlations lead to an enhancement of the typical entanglement in the sense of [8].

2. Technical introduction

We consider systems described by pairs of canonically conjugated operators $\{\hat{x}_j, \hat{p}_j\}$ with continuous spectra, acting on a tensor product of infinite dimensional Hilbert spaces. Let $\hat{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_n, \hat{p}_n)$ denote the vector of the operators $\hat{x}_j$ and $\hat{p}_j$. The canonical commutation relations for the $\hat{R}_i$ can be expressed in terms of the symplectic form Ω

$$[\hat{R}_j, \hat{R}_k] = 2i\Omega_{jk} \,,$$

$$\text{with} \quad \Omega \equiv \bigoplus_{j=1}^{n} \omega \,, \quad \omega \equiv \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \,.$$

The state of a CV system can be equivalently described by a positive trace-class operator (the density matrix ϱ) or by quasi–probability distributions. Throughout the paper, we shall focus on states with Gaussian characteristic functions and quasi-probability distributions, commonly referred to as 'Gaussian states'. By definition, a Gaussian state ϱ is completely characterised by the first and second statistical moments of the canonical operators. We will just consider states with null first moments, completely determined by the symmetric covariance matrix (CM) $\boldsymbol{\sigma}$ with entries $\boldsymbol{\sigma}_{jk} \equiv \text{Tr}\left[\varrho(\hat{X}_j \hat{X}_k + \hat{X}_k \hat{X}_j)\right]$. Being the variances and covariances of quantum operators, such entries are obtained by noise variance and noise correlation measurements (obtained by 'homodyne' detection for optical systems). They can be expressed as energies by multiplying them by the quantity $\hbar\omega$, where ω is the frequency of the considered mode. In fact, for any n-mode state the quantity $\hbar\omega\text{Tr}\left(\boldsymbol{\sigma}/4\right)$ is just the contribution of the second moments to the average of the "free" Hamiltonian $\sum_{i=1}^{n}(a_i^\dagger a_i + 1/2)$.

Let us recall some useful results about symplectic operations, along with their consequences on the description of Gaussian states. Being positive definite [9], the CM of a n–mode Gaussian state can always be written as

$$\boldsymbol{\sigma} = S^T \nu S \,, \tag{1}$$

with $S \in Sp_{(2n,\mathbb{R})}$ and

$$\nu = \text{diag}(\nu_1, \nu_1, \ldots, \nu_n, \nu_n) \,, \tag{2}$$

corresponding to the CM of a tensor product of states at thermal equilibrium with local temperatures $T_j = 2(\nu_j - 1)$. The quantities $\{\nu_j\}$ are referred to as the *symplectic eigenvalues* of the CM $\boldsymbol{\sigma}$, the transformation S is said to perform a *symplectic diagonalisation* of $\boldsymbol{\sigma}$, while the diagonal matrix with identity blocks ν is referred to as the *Williamson form* of $\boldsymbol{\sigma}$ [10]. The symplectic eigenvalues $\{\nu_j\}$ can be determined as the positive square roots of the eigenvalues of the positive matrix $-\Omega\boldsymbol{\sigma}\Omega\boldsymbol{\sigma}$. Such eigenvalues are in fact invariant under the action of symplectic transformations on the matrix $\boldsymbol{\sigma}$.

We briefly remark that all the entropic quantities of Gaussian states can be expressed in terms

of their symplectic eigenvalues. Notably, the 'purity' $\mathrm{Tr}\,\varrho^2$ of a Gaussian state ϱ is simply given by the symplectic invariant $\mathrm{Det}\,\boldsymbol{\sigma} = \prod_{i=1}^{n} \nu_i$, being $\mathrm{Tr}\,\varrho^2 = (\mathrm{Det}\,\boldsymbol{\sigma})^{-1/2}$.

Central to our analysis will also be the following general decomposition of a symplectic transformation S (referred to as the "Euler" or "Bloch-Messiah" decomposition [11, 12]):

$$S = O'ZO, \tag{3}$$

where $O, O' \in K(n) = Sp(2n, \mathbb{R}) \cap SO(2n)$ are orthogonal symplectic transformations, while

$$Z = \oplus_{j=1}^{n} \begin{pmatrix} z_j & 0 \\ 0 & \frac{1}{z_j} \end{pmatrix},$$

with $z_j \geq 1 \,\forall\, j$. The set of such Z's forms a non-compact subgroup of $Sp_{2n,\mathbb{R}}$ comprised of local (single-mode) squeezing operations (borrowing the terminology of quantum optics, where such transformations arise in degenerate parametric down-conversion processes). Moreover, let us also mention that the compact subgroup $K(n)$ is isomorphic to the unitary group $U(n)$, and is therefore characterised by n^2 independent parametres. To acquaint the reader with the flavour of the counting arguments which will accompany us through this paper (and with the nontrivial aspects contained therein), let us combine the Williamson and the Euler decomposition to determine the number of degrees of freedom of an arbitrary *mixed* n-mode Gaussian state (up to first moments), thus obtaining $n + 2n^2 + n - n = 2n^2 + n$. The first two addenda are just the sum of the number of symplectic eigenvalues (n) and of degrees of freedom of a symplectic operation ($2n^2 + n$, resulting from two symplectic orthogonal transformations and from n single-mode squeezing parametres). Finally, the subtracted n takes into account the invariance under single-mode rotations of the local Williamson forms (which 'absorbs' one degree of freedom *per mode* of the symplectic operation describing the state according to Eq. (1)). Actually, the previous result is just the number of degrees of freedom of a $2n \times 2n$ symmetric matrix (in fact, the only constraint $\boldsymbol{\sigma}$ has to fulfill to represent a physical state is the semidefinite $\boldsymbol{\sigma} + i\Omega \geq 0$, which compactly expresses the uncertainty relation for many modes [13]).

Finally, we recall the form of the CM $\boldsymbol{\sigma}^{2m}$ of a *two-mode squeezed state*:

$$\boldsymbol{\sigma}^{2m} = \begin{pmatrix} \cosh r & 0 & \sinh r & 0 \\ 0 & \cosh r & 0 & -\sinh r \\ \sinh r & 0 & \cosh r & 0 \\ 0 & -\sinh r & 0 & \cosh r \end{pmatrix}, \tag{4}$$

parametrised by the positive squeezing r. This class of states represents the prototype of CV entanglement both for the experimentalist (it can be generated by non-degenerate "parametric down conversion") and for the theorist (it encompasses, in the limit $r \to \infty$, the perfectly correlated seminal Einstein-Podolsky-Rosen state [14]) and will play a crucial role in several arguments to follow.

3. Standard forms of mixed states

Before addressing the reductions of *pure* states, let us briefly consider the standard forms of general *mixed* n-mode Gaussian states under local, single-mode symplectic operations. Let us express the CM $\boldsymbol{\sigma}$ in terms of 2×2 sub-matrices $\boldsymbol{\sigma}_{jk}$, defined by

$$\boldsymbol{\sigma} \equiv \begin{pmatrix} \boldsymbol{\sigma}_{11} & \cdots & \boldsymbol{\sigma}_{1n} \\ \vdots & \ddots & \vdots \\ \boldsymbol{\sigma}_{1n}^{\mathsf{T}} & \cdots & \boldsymbol{\sigma}_{nn} \end{pmatrix}$$

each sub-matrix describing either the local CM of mode j ($\boldsymbol{\sigma}_{jj}$) or the correlations between the pair of modes j and k ($\boldsymbol{\sigma}_{jk}$).

Let us remind the reader of the Euler decomposition of a generic single-mode symplectic transformation $S_1(\vartheta', \vartheta'', z)$:

$$S_1(\vartheta', \vartheta'', z) = \left(\begin{array}{cc} \cos\vartheta' & \sin\vartheta' \\ -\sin\vartheta' & \cos\vartheta' \end{array} \right) \left(\begin{array}{cc} z & 0 \\ 0 & \frac{1}{z} \end{array} \right) \left(\begin{array}{cc} \cos\vartheta'' & \sin\vartheta'' \\ -\sin\vartheta'' & \cos\vartheta'' \end{array} \right)$$

into two single-mode rotations ("phase shifters", with reference to the "optical phase" in phase space) and one squeezing operation. We will consider the reduction of a generic CM $\boldsymbol{\sigma}$ under local operations of the form $S_l \equiv \bigoplus_{j=1}^{n} S_1(\vartheta'_j, \vartheta''_j, z_j)$. The local symmetric blocks $\boldsymbol{\sigma}_{jj}$ can all be diagonalised by the first rotations and then symplectically diagonalised (*i.e.*, made proportional to the identity) by the subsequent squeezings, such that $\boldsymbol{\sigma}_{jj} = a_j \mathbb{1}_2$ (thus reducing the number of parametres in each diagonal block to the local symplectic eigenvalue, determining the entropy of the mode). The second series of local rotations can then be applied to manipulate the non-local blocks, while leaving the local ones unaffected (as they are proportional to the identity). Different sets of n entries in the non-diagonal sub-matrices can be thus set to zero. For an even total number of modes, all the non-diagonal blocks $\boldsymbol{\sigma}_{12}, \boldsymbol{\sigma}_{34}, \ldots, \boldsymbol{\sigma}_{(n-1)n}$ describing the correlations between disjoint pairs of quadratures can be diagonalised (leading to the singular-value diagonal form of each block), with no conditions on all the other blocks. For an odd number of modes, after the diagonalisation of the blocks relating disjoint quadratures, a further non-diagonal block involving the last mode (say, $\boldsymbol{\sigma}_{1n}$) can be put in triangular form by a rotation on the last mode.

Notice finally that the locally invariant degrees of freedom of a generic Gaussian state of n modes are $(2n+1)n - 3n = 2n^2 - 2n$, as follows from the subtraction of the number of free parametres of the local symplectics from the one of a generic state – with an obvious exception for $n = 1$, for which the number of free parametres is 1, due to the rotational invariance of single-mode Williamson forms (see the discussion about the vacuum state in Sec. 5).

4. Degrees of freedom of pure Gaussian states

Pure Gaussian states are characterised by CMs with Williamson form equal to the identity. As we have seen, the Williamson decomposition provides a mapping from any Gaussian state into the uncorrelated product of thermal (generally mixed) states: such states are pure (corresponding to the vacuum), if and only if all the symplectic eigenvalues are equal to 1.

The symplectic eigenvalues of a generic CM $\boldsymbol{\sigma}$ are determined as the eigenvalues of the matrix $|i\Omega\boldsymbol{\sigma}|$, where Ω stands for the symplectic form. Therefore, a Gaussian state of n-modes with CM $\boldsymbol{\sigma}$ is pure if and only if

$$-\boldsymbol{\sigma}\Omega\boldsymbol{\sigma}\Omega = \mathbb{1}_{2n} . \tag{5}$$

It will be convenient here to reorder the CM, and to decompose it in the three sub-matrices $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_p$ and $\boldsymbol{\sigma}_{xp}$, whose entries are defined as

$$(\boldsymbol{\sigma}_x)_{jk} = \text{Tr}\left[\varrho\hat{x}_j\hat{x}_k\right] , \quad (\boldsymbol{\sigma}_p)_{jk} = \text{Tr}\left[\varrho\hat{p}_j\hat{p}_k\right] , \quad (\boldsymbol{\sigma}_{xp})_{jk} = \text{Tr}\left[\varrho\{\hat{x}_j, \hat{p}_k\}/2\right] , \tag{6}$$

such that the complete CM $\boldsymbol{\sigma}$ is given in block form by

$$\boldsymbol{\sigma} = \left(\begin{array}{cc} \boldsymbol{\sigma}_x & \boldsymbol{\sigma}_{xp} \\ \boldsymbol{\sigma}_{xp}^{\mathsf{T}} & \boldsymbol{\sigma}_p \end{array} \right) . \tag{7}$$

Let us notice that the matrices $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_p$ are always symmetric and strictly positive, while the matrix $\boldsymbol{\sigma}_{xp}$ does not obey any general constraint.

Eqs. (5) and (7) straightforwardly lead to the following set of conditions

$$\boldsymbol{\sigma}_x \boldsymbol{\sigma}_p = \mathbb{1}_n + \boldsymbol{\sigma}_{xp}^2 \;, \tag{8}$$

$$\boldsymbol{\sigma}_{xp}\boldsymbol{\sigma}_x - \boldsymbol{\sigma}_x \boldsymbol{\sigma}_{xp}^\mathsf{T} = 0 \;, \tag{9}$$

$$\boldsymbol{\sigma}_p \boldsymbol{\sigma}_x = \mathbb{1}_n + \boldsymbol{\sigma}_{xp}^{\mathsf{T}\,2} \;, \tag{10}$$

$$\boldsymbol{\sigma}_{xp}^\mathsf{T}\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_p \boldsymbol{\sigma}_{xp} = 0 \;. \tag{11}$$

Now, Eq. (10) is obviously obtained by transposition of Eq. (8). Moreover, from (8) one gets

$$\boldsymbol{\sigma}_p = \boldsymbol{\sigma}_x^{-1}(\mathbb{1}_n + \boldsymbol{\sigma}_{xp}^2) \;, \tag{12}$$

while Eq. (9) is equivalent to

$$\boldsymbol{\sigma}_x^{-1}\boldsymbol{\sigma}_{xp} - \boldsymbol{\sigma}_{xp}^\mathsf{T}\boldsymbol{\sigma}_x^{-1} = 0 \tag{13}$$

(the latter equations hold generally, as $\boldsymbol{\sigma}_x$ is strictly positive and thus invertible). Eq. (13) allows one to show that any $\boldsymbol{\sigma}_p$ determined by (12) satisfies the condition (11). Therefore, only Eqs. (8) and (9) constitute independent constraints and fully characterise the CM of pure Gaussian states.

Given any (strictly positive) matrix $\boldsymbol{\sigma}_x$ and (general) matrix $\boldsymbol{\sigma}_{xp}$, the fulfillment of condition (9) allows to specify the second moments of any pure Gaussian state, whose sub-matrix $\boldsymbol{\sigma}_p$ is determined by Eq. (12) and does not involve any additional degree of freedom.

A straightforward counting argument thus yields the number of degrees of freedom of an arbitrary pure Gaussian state‡, by adding the entries of a general and of a symmetric $n \times n$ matrices and subtracting the equations of the antisymmetric condition (9): $n^2 + n(n+1)/2 - n(n-1)/2 = n^2 + n$, in compliance with the number dictated by the Euler decomposition of a symplectic operation:

$$\boldsymbol{\sigma} = S^\mathsf{T} \mathbb{1}_{2n} S = O^\mathsf{T} Z^2 O \;. \tag{14}$$

Notice that, if either $\boldsymbol{\sigma}_x$ or $\boldsymbol{\sigma}_{xp}$ are kept fixed, the constraint (9) is just a linear constraint on the entries of the other matrix, which can always be solved (it cannot be overdetermined, since the number of equations $n(n-1)/2$ is always smaller than the number of variables, either n^2 or $n(n+1)/2$).

A preliminary insight into the role of local operations in determining the number of degrees of freedom of pure CMs is gained by analysing the counting of free parametres in the continuous variable analogue of the Schmidt decomposition. The CM of any pure $(m + n)$-mode Gaussian state is equivalent, up to local symplectic transformations on the m-mode and n-mode subsystems, to the tensor product of m decoupled two-mode squeezed states (assuming, without loss of generality, $m < n$) and $n - m$ uncorrelated vacua [16]. Besides the m two-mode squeezing parametres, the degrees of freedom of the local symplectic transformations to be added are $2n^2 + n + 2m^2 + m$. However, a mere addition of these two values leads to an overestimation with respect to the number of free parametres of pure CMs determined above. This is due to the invariance of the CM in 'Schmidt form' under specific classes of local operations. Firstly, the $(n - m)$-mode vacuum (with CM equal to the identity) is trivially invariant under local orthogonal symplectics, which account for $(n - m)^2$ parametres. Furthermore, one parametre is lost for each two-mode squeezed block with CM $\boldsymbol{\sigma}^{2m}$ given by Eq. (4): this is due to an invariance under single-mode rotations peculiar to two-mode squeezed states. For such states, the sub-matrices $\boldsymbol{\sigma}_x^{2m}$ and $\boldsymbol{\sigma}_p^{2m}$ have identical – and all equal – diagonal entries, while the sub-matrix $\boldsymbol{\sigma}_{xp}^{2m}$ is null. Local rotations embody two

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‡ The same number could have been inferred, via an essentially identical reasoning, from the normal form of pure Gaussian states independently derived in Ref. [15], Lemma 1.

degrees of freedom – two local 'angles' in phase space – in terms of operations. Now, because they act locally on 2×2 identities, rotations on both single modes cannot affect the diagonals of $\boldsymbol{\sigma}_x^{2m}$ and $\boldsymbol{\sigma}_p^{2m}$, nor the diagonal of $\boldsymbol{\sigma}_{xp}^{2m}$, which is still equal to zero. In principle, they could thus lead to two (possibly different) non-diagonal elements for $\boldsymbol{\sigma}_{xp}^{2m}$ and/or to two different non-diagonal elements for $\boldsymbol{\sigma}_x^{2m}$ and $\boldsymbol{\sigma}_p^{2m}$ (which, at the onset, have opposite non-diagonal elements, see Eq. (4)), resulting in

$$\boldsymbol{\sigma}_x^{2m} = \left(\begin{array}{cc} a & c_1 \\ c_1 & a \end{array} \right) , \boldsymbol{\sigma}_p^{2m} = \left(\begin{array}{cc} a & c_2 \\ c_2 & a \end{array} \right) , \boldsymbol{\sigma}_{xp}^{2m} = \left(\begin{array}{cc} 0 & y \\ z & 0 \end{array} \right)$$

However, elementary considerations, easily worked out for such 2×2 matrices, show that Eqs. (9) and (12) imply

$$c_1 = -c_2 \ , \ \ y = z \ \ \text{and} \ \ a^2 - c_1^2 = 1 + y^2 \,.$$

These constraints reduce from five to two the number of free parametres in the state: *the action of local single-mode rotations – generally embodying two independent parametres – on two-mode squeezed states, allows for only one further independent degree of freedom.* In other words, all the Gaussian states resulting from the manipulation of two-mode squeezed states by local rotations ("phase-shifters", in the experimental terminology) can be obtained by acting on only one of the two modes. One of the two degrees of freedom is thus lost and the counting argument displayed above has to be recast as $m+2n^2+n+2m^2+m-(m-n)^2-m = (m+n)^2 + (m+n)$, in compliance with what we had previously established.

As we are about to see, this invariance, peculiar to two-mode squeezed states, also accounts for the reduction of locally invariant free parametres occurring in pure two-mode Gaussian states.

5. Reduction under single-mode operations

Let us now determine the reduction of degrees of freedom achievable for pure Gaussian states by applying local single-mode symplectic transformations. Notice that all the entanglement properties (both bipartite and multipartite) of the states will solely depend on the remaining parametres, which cannot be canceled out by LU operations.

In general, for n-mode systems, local symplectic operations have $3n$ degrees of freedom, while n-mode pure Gaussian states are specified, as we just saw, by $n^2 + n$ quantities. The subtraction of these two values yields a residual number of parametres equal to $n^2 - 2n$. However, this number holds for $n \geq 3$, but fails for single- and two-mode states. Let us analyse the reasons of this occurrence.

For single-mode systems, the situation is trivial, as one is allowing for all the possible operations capable, when acting on the vacuum, to unitarily yield any possible state. The number of free parametres is then clearly zero (as any state can be reduced into the vacuum state, with CM equal to the 2×2 identity). The expression derived above would instead give -1. The reason of this mismatch is just to be sought in the invariance of the vacuum under local rotations: only two of the three parametres entering the Euler decomposition actually affect the state. On the other hand, one can also notice that these two latter parametres, characterising the squeezing and subsequent last rotation of the Euler decomposition acting on the vacuum, are apt to completely reproduce any possible single-mode state. Clearly, this situation is the same as for any n-mode pure Gaussian state under global operations: the first rotation of the Euler decomposition is always irrelevant, thus implying a corresponding

reduction of the free parametres of the state with respect to the most general symplectic operation.

As for two-mode states, the counting argument above would give zero locally invariant parametres. On the other hand, the existence of a class of states with a continuously varying parametre determining the amount of bipartite entanglement [the two-mode squeezed states of Eq. (4)], clearly shows that the number of free parametres cannot be zero. Actually, local symplectic operations allows one to bring *any* (pure or mixed) two-mode Gaussian state in a "standard form" with $\boldsymbol{\sigma}_{xp} = 0$ and with identical diagonals for $\boldsymbol{\sigma}_x$ and $\boldsymbol{\sigma}_p$. Imposing then Eq. (9) on such matrices, one finds that the only pure states of such a form have to be two-mode squeezed states. Therefore, we know that the correct number of locally invariant free parametres has to be one. Even though local symplectic operations on two-mode states are determined by 6 parametres, they can only cancel 5 of the 6 parametres of pure two-mode states. This is, again, due to the particular transformation properties of two-mode squeezed states under single-mode rotations, already pointed out in Section 4 when addressing the counting of degrees of freedom in the Schmidt-like decomposition: local rotations acting on a two-mode squeezed state add only one independent parametre. The most general two-mode pure Gaussian state results from a two-mode squeezed state by a single local rotation on any of the two modes, followed by two local squeezings and two further rotations acting on different modes. Notice that the same issue arises for $(m + n)$-mode states to be reduced under local m- and n-mode symplectic operations. A mere counting of degrees of freedom would give a residual number of local free parametres equal to $(m + n)^2 + m + n - 2m^2 - 2n^2 - m - n = -(m - n)^2$. This result is obviously wrong, again due to a loss of parametres in the transformations of particular invariant states. We have already inspected this very case and pointed out such invariances in our treatment of the Schmidt decomposition (previous Section): we know that the number of locally irreducible free parametres is just $\min(m, n)$ in this case, corresponding to the tensor product of two-mode squeezed states and uncorrelated vacua.

For $n \geq 3$, local single-mode operations can fully reduce the number of degrees of freedom of pure Gaussian states by their total number of parametres. The issue encountered for two-mode states does not occur here, as the first single-mode rotations can act on different non-diagonal blocks of the CM (*i.e.*, pertaining to the correlations between different pairs of modes). The number of such blocks is clearly equal to $(n^2 - n)/2$ while the number of local rotations is just n. Only for $n = 1, 2$ is the latter value larger than the former: this is, ultimately, why the simple subtraction of degrees of freedom only holds for $n \geq 3$. To better clarify this point, let us consider a CM $\boldsymbol{\sigma}^{3m}$ in the limiting instance $n = 3$. The general standard form for (mixed) three-mode states implies the conditions (see Sec. 3)

$$\mathrm{diag}\,(\boldsymbol{\sigma}_x^{3m}) = \mathrm{diag}\,(\boldsymbol{\sigma}_p^{3m}) \tag{15}$$

and

$$\boldsymbol{\sigma}_{xp}^{3m} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & u \\ s & t & 0 \end{pmatrix}\,. \tag{16}$$

The diagonal of $\boldsymbol{\sigma}_x^{3m}$ coincides with that of $\boldsymbol{\sigma}_p^{3m}$ (which always results from the local single-mode Williamson reductions) while six entries of $\boldsymbol{\sigma}_{xp}^{3m}$ can be set to zero. For pure states, imposing Eq. (9) results into a linear system of three equations for the nonzero entries of $\boldsymbol{\sigma}_{xp}^{3m}$, with coefficients given by the entries of $\boldsymbol{\sigma}_x^{3m}$. Exploiting the complete positivity of $\boldsymbol{\sigma}_x^{3m}$, one can show that such a system implies $s = t = u = 0$. Therefore, for pure three-mode Gaussian states, the matrix $\boldsymbol{\sigma}_{xp}^{3m}$ can be set to zero by local symplectic operations alone on the individual modes. The entries of the symmetric positive definite matrix $\boldsymbol{\sigma}_x^{3m}$ are constrained

by the necessity of Eqs. (8) – which just determines $\boldsymbol{\sigma}_p^{3m}$ – and (15), which is comprised of three independent conditions and further reduces the degrees of freedom of the state to the predicted value of three. An alternative proof of this is presented in Ref. [17].

Let us also incidentally remark that the possibility of reducing the sub-matrix $\boldsymbol{\sigma}_{xp}$ to zero by local single-mode operations is exclusive to two-mode (pure and mixed) and to three-mode pure states. This is because, for general Gaussian states, the number of parametres of $\boldsymbol{\sigma}_{xp}$ after the local Williamson diagonalisations is given by $n(n-1)$ (two per pair of modes) and only n of these can be canceled out by the final local rotations, so that only for $n < 3$ can local operations render $\boldsymbol{\sigma}_{xp}$ null. For pure states and $n > 2$ then, further $n(n-1)/2$ constraints on $\boldsymbol{\sigma}_{xp}$ ensue from the antisymmetric condition (9): this number turns out to match the number of free parametres in $\boldsymbol{\sigma}_{xp}$ for $n = 3$, but it is no longer enough to make $\boldsymbol{\sigma}_{xp}$ null for pure states with $n \geq 4$.

Summing up, we have rigorously determined the number of "locally irreducible" free parametres of pure Gaussian states, unambiguously showing that the quantification and qualification of the entanglement (which, by definition, is preserved under LU operations) in such states of n modes is completely determined by 1 parametre for $n = 2$ and $(n^2 - 2n)$ parametres for $n > 2$.

6. Efficient state engineering of multimode pure Gaussian states

It would be desirable to associate the mathematically clear number $(n^2 - 2n)$ with an operational, physical insight. In other words, it would be useful for experimentalists (working, for instance, in quantum optics) to be provided with a recipe to create pure n-mode Gaussian states with completely general entanglement properties in an 'economical' way, in the precise, specific sense that exactly $(n^2 - 2n)$ optical elements are used. A transparent approach to develop such a procedure consists in considering the reverse of the phase space $1 \times (n - 1)$ Schmidt decomposition, as introduced in Section 4. Namely, a completely general (not accounting for the local invariances) state engineering prescription for pure Gaussian states can be cast in two main steps: (i) create a two-mode squeezed state of modes 1 and 2, which corresponds to the multimode state in its Schmidt form; (ii) operate with the most general $(n-1)$-mode symplectic transformation S^{-1} on the block of modes $\{2, 3, \ldots, n\}$ (with modes $i = 3, \ldots, n$ initially in the vacuum state) to redistribute entanglement among all modes. The operation S^{-1} is the inverse of the transformation S which brings the reduced CM of modes $\{2, 3, \ldots, n\}$ in its Williamson diagonal form. It is also known that any such symplectic transformation S^{-1} (unitary on the Hilbert space) can be decomposed in a network of optical elements [18].§ The number of elements required to accomplish this network, however, will in general greatly exceed the minimal number of parametres on which the entanglement between any two sub-systems depends. Shifting the LU optimisation from the final CM, back to the engineering symplectic network, is in principle an extremely involuted and nontrivial task.

This problem has been solved in Ref. [7] for a special subclass of Gaussian states, which is of null measure but still of central importance for practical implementations. It is constituted by those pure n-mode Gaussian states which can be locally put in a standard form with null $\boldsymbol{\sigma}_{xp}$. This class encompasses generalised GHZ-type Gaussian states, useful for CV quantum teleportation networks [19], and Gaussian cluster states [20] employed in CV implementations of one-way quantum computation [21]. It also comprises (as proven in the previous Section) all three-mode pure Gaussian states [17], whose usefulness for CV quantum communication

§ Notice that, even though Ref. [18] refers to compact ('*passive*') transformations alone, the Euler decomposition, which involves only passive operations and single-mode squeezings, allows one to straightforwardly extend such decompositions in terms of single- and two-mode operations to general symplectic transformations.

purposes has been thoroughly investigated [22]. In the physics of many-body systems, those states are quite ubiquitous as they are ground states of harmonic Hamiltonians with spring-like interactions [23]. For these Gaussian states, which we shall call here *block-diagonal*, the minimal number of LU-invariant parametres reduces to $n(n-1)/2$ for any n.‖ Accordingly, one can show that an efficient scheme can be devised to produce block-diagonal pure Gaussian states, involving exactly $n(n-1)/2$ optical elements which in this case are only constituted by single-mode squeezers and beam-splitters, in a given sequence [7].

Borrowing the ideas leading to the state engineering of block-diagonal pure Gaussian states, we propose here a scheme, involving $(n^2 - 2n)$ independent optical elements, to produce more general n-mode pure Gaussian states encoding correlations between positions and momentum operators as well. To this aim, we introduce 'counter-beam splitter' transformations, named "*seraphiques*", which, recovering the phase space ordering of Sec. 2, act on two modes j and k as

$$
C_{j,k}(\vartheta) = \begin{pmatrix} \cos(\vartheta) & 0 & 0 & \sin(\vartheta) \\ 0 & \cos(\vartheta) & -\sin(\vartheta) & 0 \\ 0 & \sin(\vartheta) & \cos(\vartheta) & 0 \\ -\sin(\vartheta) & 0 & 0 & \cos(\vartheta) \end{pmatrix}.
$$

Such operations can be obtained by usual beam splitters (which we will denote by $B_{j,k}(\vartheta)$) by applying a $\pi/2$ phase shifter P_k on *only one* of the two considered modes. P_k is a local rotation mapping, in Heisenberg picture, $\hat{x}_k \mapsto -\hat{p}_k$ and $\hat{p}_k \mapsto \hat{x}_k$. In phase space, one has $C_{j,k}(\vartheta) = P_k^\mathsf{T} B_{j,k}(\vartheta) P_k$. Notice that, even though $C_{j,k}(\vartheta)$ is equal to the product of single-mode operations and beam splitters, this does not mean that such a transformation is "equivalent" to a beam splitter in terms of state generation. In fact, the local operations do not commute with the beam splitters, so that a product of the kind $B_{j,k}(\vartheta')C_{j,k}(\vartheta'')$ *cannot* be written as $B_{j,k}(\vartheta)S_l$ for some local operation S_l and ϑ.

The state engineering scheme runs along the lines as the one for the block-diagonal states, the only modification being that for each pair of modes except the last one $(n-1, n)$, a beam-splitter transformation is followed by a seraphique. In more detail (see Fig. 1): first of all (step i), one squeezes mode 1 of an amount s, and mode 2 of an amount $1/s$ (*i.e.* one squeezes the first mode in one quadrature and the second, of the same amount, in the orthogonal quadrature); then one lets the two modes interfere at a $50:50$ beam splitter. One has so created a two-mode squeezed state between modes 1 and 2, which corresponds to the Schmidt form of the pure Gaussian state with respect to the $1 \times (n-1)$ bipartition. The second step basically corresponds to a re-distribution, or allotment, of the initial two-mode entanglement among all modes. This task can be obtained by letting each additional mode interact step-by-step with all the previous ones, via beamsplitters and seraphiques (which are in turn combinations of beam splitters and phase shifters). Starting with mode 3 (which was in the vacuum like all the subsequent ones), one thus squeezes it (by an amount r_3) and combines it with mode 2 via a beam-splitter (characterized by a transmittivity $b_{2,3}$) and a subsequent seraphique (parametrised by $c_{2,3}$). Then one squeezes mode 4 by r_4 and lets it interfere sequentially both with mode 2 (via a beamsplitter with $b_{2,4}$ and a seraphique with $c_{2,4}$) and with mode 3 ($b_{3,4}$ and $c_{3,4}$). This process can be iterated for each other mode, as shown in Fig. 1, until the last mode n is squeezed (r_n) and entangled with the previous ones via beam-splitters with respective transmittivities $b_{i,n}$, $i = 2, \cdots, n-1$, and corresponding seraphiques

---

‖ This number is easily derived from the general framework developed in Sec. 4: for $\boldsymbol{\sigma}_{xp} = 0$, Eqs. (8) and (9) reduce to $\boldsymbol{\sigma}_x = \boldsymbol{\sigma}_p^{-1}$. The only further condition to impose after the local reduction is then $\mathrm{diag}(\boldsymbol{\sigma}_x) = \mathrm{diag}(\boldsymbol{\sigma}_x^{-1})$, which brings the number of free parametres of the symmetric $\boldsymbol{\sigma}_x$ from $(n+1)n/2$ down to $n(n-1)/2$

with amplitudes $c_{i,n}$, $i = 2, \cdots, n - 2$. We remark that mode 1 becomes entangled with all the other modes as well, even if it never comes to a direct interaction with each of modes $3, \cdots, n$.
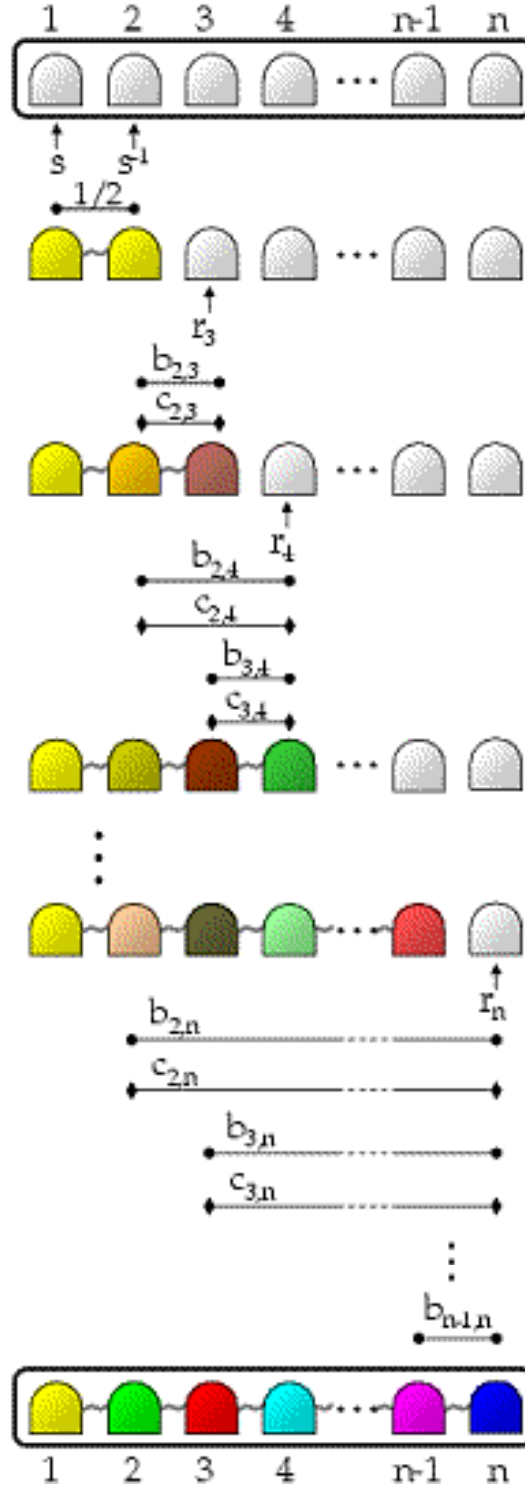
Figure 1. (Color online) Possible scheme to create a general n-mode pure Gaussian state. White shaded door-shaped (*Schlutzkrapfen*-like) objects depict vacuum modes, while each color corresponds to a different single-mode determinant (*i.e.* different degrees of local mixedness). Vertical arrows denote single-mode squeezing operations with squeezing parametres r_j, horizontal circle-ended lines denote beam-splitting operations $b_{i,j}$ between modes i and j, and horizontal diamond-ended lines denote two-mode seraphiques parametrised by $c_{i,j}$. See text for details.

This scheme is implemented with minimal resources. Namely, the state engineering process is characterised by one squeezing degree (step i), plus $n - 2$ individual squeezings, together with $\sum_{i=1}^{n-2} i = (n-1)(n-2)/2$ beam-splitter transmittivities, and $[\sum_{i=1}^{n-2} i] - 1 = n(n-3)/2$ seraphique transmittivities, which amount to a total of $(n^2 - 2n)$ quantities, exactly the ones parametrising a general pure Gaussian state of $n \geq 3$ modes up to local symplectic operations. While this scheme is surely more general than the one for block-diagonal states, as it enables to efficiently create a broader class of pure Gaussian states for $n > 3$, we shall leave it as an open question to check if the recipe of Fig. 1 is general enough to produce *all* pure n-mode Gaussian states up to LUs. Verifying this analytically leads to pretty cumbersome expressions already for $n = 4$. Instead, it would be very interesting to investigate if the average entanglement of the output Gaussian states numerically obtained by a statistically significant sample of applications of our scheme with random parametres, matches the *typical* entanglement of pure Gaussian states under "thermodynamical" state-space measures as computable along the lines of Ref. [8]. This would prove the optimality and generality of our scheme in an operational way, which is indeed more useful for practical applications.

7. Epilogue

In view of the previous, comprehensive characterisation of structural and informational properties of pure n-mode Gaussian states under LU operations, it is natural to question if the $n(n - 3)/2$ additional parametres encoded in $\hat{x}$-$\hat{p}$ correlations for non-block-diagonal states, have a definite impact on the bipartite and multipartite entanglement.

At present, usual CV protocols are devised, even in multimode settings, to make use of states without any $\hat{x}$-$\hat{p}$ correlations. In such cases, the economical (relying on $(n - 1)n/2$ parametres) "block-diagonal state engineering" scheme detailed in Ref. [7] is clearly the optimal general strategy for the production of entangled resources. However, theoretical considerations strongly suggest that states with $\boldsymbol{\sigma}_{xp} \neq 0$ might have remarkable potential for improved quantum-informational applications. In fact, considering again the thermodynamical entanglement framework of Gaussian states [8], one can define natural averages either on the whole set of pure Gaussian states, or restricting to states with $\boldsymbol{\sigma}_{xp} = 0$.¶ Well, numerics unambiguously show (see Fig. 2) that the average entanglement (under any bipartition) of Gaussian states without $\hat{x}$-$\hat{p}$ correlations (like the ones considered in [7]) is systematically lower than the typical entanglement of more general pure Gaussian states, with this behaviour getting more and more manifest as the total number of modes increases (clearly, according to what we have shown in this work, this only occurs for $n > 3$). In a way, the full entanglement potential of Gaussian states is diminished by the restriction to block-diagonal states.

On the other hand, the comparison between the average entanglement generated in randomising processes based on the engineering scheme proposed here and the block diagonal one is under current investigation as well. If the present scheme turned out to be out-performing the previous ones in terms of entanglement generation – as expected in view of the argument above – this would be a spur to the exploration of novel CV protocols, capable of adequately exploiting $\hat{x}$-$\hat{p}$ correlated resources.

¶ The average over the whole set of pure Gaussian states is realised by integrating over the Haar measure of the compact subgroup $K(n)$, isomorphic to $U(n)$. The restriction to all the states with vanishing xp block is instead achieved by considering only orthogonal symplectic transformations of the form $R \oplus R$ with $R \in O(n)$ – which form a group isomorphic to $O(n)$ – and by integrating over the Haar measure of $O(n)$ [as opposed to $U(n)$].
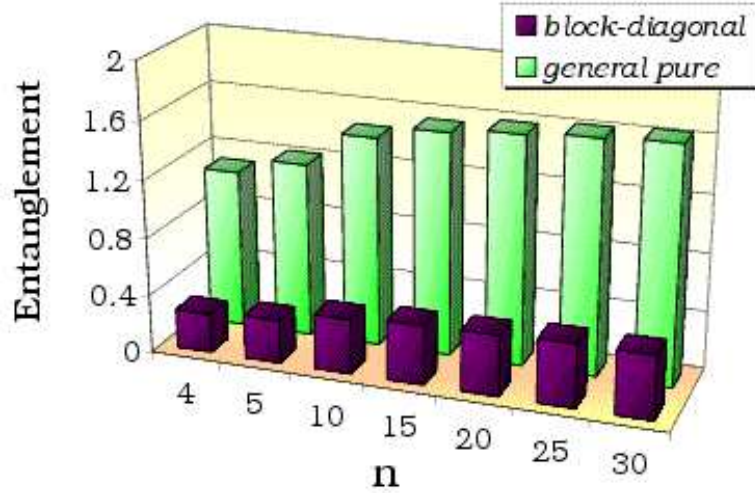
Figure 2. (Color online) Typical entanglement, measured by the Von Neumann entropy, between one mode and the remaining $n-1$ modes in two classes of pure n-mode Gaussian states, for $n \geq 4$. Light green bars denote completely general pure states, while dark purple bars refer to block-diagonal pure states. For each n, the entanglement is averaged over 10000 random realisations of pure Gaussian states (with and without direct $\hat{x}$-$\hat{p}$ correlations, respectively) according to the microcanonical state space measure introduced in [8], at a fixed total energy $\mathcal{E} = 5n$. Nonvanishing correlations between position and momentum operators in the covariance matrix, clearly yield an increase in the typical entanglement of pure Gaussian states, more evident with increasing number n of modes.

Acknowledgments

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